MINIMIZING THE RISK
OF A FINANCIAL PRODUCT USING A PUT OPTION.

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Abstract

In this paper, we elaborate a formula for determining the optimal strike price for a put option, used to hedge a position in a financial product such as an asset or a bond. This strike price is optimal in the sense that it minimizes, for a given budget, a risk measure such as Value-at-Risk or Tail Value-at-Risk. Formulas are derived for one single underlying as well as for a weighted average of underlyings which can also be understood as a portfolio. For the latter we will consider two cases depending on the dependence structure of the components in the weighted sum.

1. INTRODUCTION

The importance of a sound risk management system can hardly be underestimated. The advent of new capital requirements for both the banking (Basel II) and insurance (Solvency II) industry, are two recent examples of the growing concern of regulators for the financial health of firms in the economy. This paper adds to this goal. In particular, we consider the problem of determining the optimal strike price for a put option, which is used to hedge the risk of an investment in a risky financial product, such as a share, a basket of assets, a zero-coupon or a coupon-bearing bond. In order to measure risk, we consider a general risk measure and focus in particular on Value-at-Risk and Tail Value-at-Risk. The optimization is constrained by a maximum hedging budget. Alternatively, our approach can also be used to determine the minimal budget that a firm needs to spend in order to achieve a predetermined absolute risk level. For an investment in a portfolio of risky assets we are dealing with a sum of risks as well as with a put option on multiple underlyings. When the risks are comonotone

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the risk minimizing problem can easily be dealt with. When the risks are not comonotone we propose an approximation of the problem. We illustrate our method for an investment in a coupon-bearing bond where the instantaneous-short-rate is modelled by the two-additive-factor Gaussian model G2++ (see Brigo and Mercurio (2001)).

The current paper can be seen as an extension of Ahn et al. (1999), Deelstra et al. (2006) and Annaert et al. (2006), who consider the particular problems for an investment in one share and in a bond for which the instantaneous-short-rate model satisfies the assumption to apply the Jamshidian decomposition.

The paper is composed as follows: In section 2 we recall the basic notions of comonotonicity and convex ordering. In section 3 we introduce some well-known risk measures and their properties. The hedging problem is formulated in section 4 as a constrained risk minimization problem. First we deal with the case of one risky asset. In section 5 we look at the case of a portfolio of risky assets. We finish with an application in section 6.

2. COMONOTONICITY

We shortly introduce the concepts of convex order and comonotonicity. For more details and proofs of the reported results we refer the reader to the overview papers Dhaene et al. (2002a) and Dhaene et al. (2002b).

Definition 2.1 *A random variable X is said to precede a random variable Y in the convex order sense, notation $X \leq_{cx} Y$, if the following conditions hold:*

$$E[X] = E[Y],$$
$$E\left[(X - d)_+\right] \leq E\left[(Y - d)_+\right], \quad \text{for all } d.$$

The inverse of a cumulative distribution function (cdf) is usually defined as follows:

Definition 2.2 *The inverse of the cumulative distribution function F_X of a random variable X is given by*

$$F_X^{-1}(p) = \inf \left\{ x \in \mathbb{R} \mid F_X(x) \geq p \right\}, \quad p \in [0, 1]. \tag{1}$$

Next, we define comonotonicity of a random vector.

Definition 2.3 *A random vector $(Y_1, \ldots, Y_n)$ with marginal cdfs $F_{Y_i}(x) = \Pr[Y_i \leq x]$ is said to be comonotonic if it has the same distribution as $\left(F_{Y_1}^{-1}(U), F_{Y_2}^{-1}(U), \ldots, F_{Y_n}^{-1}(U)\right)$, with U a random variable which is uniformly distributed on the unit interval $(0, 1)$.*

The components of the comonotonic random vector $\left(F_{Y_1}^{-1}(U), F_{Y_2}^{-1}(U), \ldots, F_{Y_n}^{-1}(U)\right)$ are maximally dependent in the sense that all of them are non-decreasing functions of the same random variable. The following characterisation also holds:

Property 2.1 *A random vector $(Y_1, \ldots, Y_n)$ is said to be comonotonic, if there exist a random variable Z and non-decreasing (or either non-increasing) functions $g_1, g_2, \ldots, g_n \colon \mathbb{R} \to \mathbb{R}$ such that*

$$(Y_1, Y_2, \ldots, Y_n) \stackrel{d}{=} (g_1(Z), g_2(Z), \ldots, g_n(Z)),$$

where the notation $\stackrel{d}{=}$ stands for 'equality in distribution'.

Consider a random vector $(Y_1, \ldots, Y_n)$. Its comonotonic counterpart $(Y_1^c, \ldots, Y_n^c)$ is a comonotonic random vector with the same marginal distributions:

$$(Y_1^c, \ldots, Y_n^c) \overset{d}{=} \left(F_{Y_1}^{-1}(U), F_{Y_2}^{-1}(U), \ldots, F_{Y_n}^{-1}(U) \right). \tag{2}$$

The sum of the components of $(Y_1^c, \ldots, Y_n^c)$ is denoted by S^c,

$$S^c = Y_1^c + \cdots + Y_n^c. \tag{3}$$

The distribution function of S^c is completely specified when the marginals F_{Y_i} are given. The probabilities $F_{S^c}(x)$ follow from

$$F_{S^c}(x) = \sup \left\{ p \in [0,1] \mid \sum_{i=1}^{n} F_{Y_i}^{-1}(p) \leq x \right\}. \tag{4}$$

The sum S^c in (3) is in convex order sense larger than the sum $S = Y_1 + \cdots + Y_n$ while the sum S^ℓ defined as

$$S^\ell = \mathrm{E}[S|\Lambda] = \sum_{i=1}^{n} \mathrm{E}[Y_i|\Lambda] \tag{5}$$

for some conditioning random variable Λ is smaller than S in convex order sense, i.e.

$$S^\ell \leq_{cx} S \leq_{cx} S^c. \tag{6}$$

The vector

$$(\mathrm{E}[Y_1|\Lambda], \ldots, \mathrm{E}[Y_n|\Lambda]) \tag{7}$$

is not necessarily comonotonic and hence the sum S^ℓ is not necessarily a comonotonic sum. To be comonotonic, the conditioning random variable should be carefully chosen such that Property 2.1 is satisfied.

3. RISK MEASURES

3.1. Risk measure

Consider a set Γ of real valued random variables defined on a given probability space $(\Omega, \mathcal{F}, P)$. We assume that $Y_1, Y_2 \in \Gamma$ implies that $Y_1 + Y_2 \in \Gamma$, and also $aY_1 \in \Gamma$ for any $a > 0$ and $Y_1 + b \in \Gamma$ for any b. Any function $\rho : \Gamma \to \mathbb{R}$ that assigns a real number to any element of Γ is called a *risk measure* (with domain Γ).

Properties of risk measures have been investigated extensively, see e.g. Artzner et al. (1999). Some well-known properties that risk measures may or may not satisfy are monotonicity, positive homogeneity, translation invariance, subadditivity and additivity for comonotonic risks. They are defined as follows:

- *Monotonicity*: for any $Y_1, Y_2 \in \Gamma$, one has that $Y_1 \leq Y_2$ implies $\rho[Y_1] \leq \rho[Y_2]$.

- *Positive homogeneity*: for any $Y \in \Gamma$ and $a > 0$, one has that $\rho[aY] = a\rho[Y]$.

- *Translation invariance*: for any $Y \in \Gamma$ and $b \in \mathbb{R}$, one has that $\rho[Y + b] = \rho[Y] + b$.

- *Subadditivity*: for any $Y_1, Y_2 \in \Gamma$, one has that $\rho[Y_1 + Y_2] \le \rho[Y_1] + \rho[Y_2]$.

- *Additivity of comonotonic risks*: for any $Y_1, Y_2 \in \Gamma$ which are comonotonic, one has that $\rho[Y_1 + Y_2] = \rho[Y_1] + \rho[Y_2]$.

In Artzner et al. (1999) a risk measure that satisfies the properties of monotonicity, positive homogeneity, translation invariance and most (noticeably) subadditivity is called a *coherent* risk measure.

Note that for the translation invariance property, we have that adding an amount increases the risk since we consider in what follows Y as a loss when $Y > 0$.

3.2. Some well-known risk measures

With the notations of Dhaene et al. (2002a), Dhaene et al. (2003) and Dhaene et al. (2006) we recall some well-known risk measures frequently used in financial and actuarial literature.

A very popular risk measure is the p-quantile risk measure, often called the Value-at-Risk (VaR) at level p. For any p in $(0, 1)$, the p-quantile risk measure for a random variable Y, which will be denoted by $Q_p[Y]$, is defined by

$$Q_p[Y] = \inf\{x \in \mathbb{R} \mid F_Y(x) \ge p\}, \qquad p \in (0, 1), \qquad (8)$$

where $F_Y(x) = \Pr[Y \le x]$ is the cumulative distribution function (cdf) of Y taken under the same measure as in which the risk measure is considered. We also introduce the risk measure $Q_p^+[Y]$ which is defined by

$$Q_p^+[Y] = \sup\{x \in \mathbb{R} \mid F_Y(x) \le p\}, \qquad p \in (0, 1). \qquad (9)$$

Note that only values of p corresponding to a horizontal segment of F_Y lead to different values of $Q_p[Y]$ and $Q_p^+[Y]$. A case that can occur when the cdf F_Y is not strictly increasing.

By the convention $\inf \mathbb{R} = -\infty$ the quantile for $p = 0$ follows from expression (8): $Q_0[Y] = -\infty$. For a bounded random variable Y, we find in accordance with (8) that $Q_1[Y] = \max(Y)$.

Further we note that $Q_p[Y]$ is nothing else but $F_Y^{-1}(p)$ cfr. (1), while $Q_p^+[Y]$ is often denoted by $F_Y^{-1+}(p)$. The quantile function $Q_p[Y]$ is a non-decreasing and left-continuous function of p.

Although frequently used, this single quantile risk measure of a predetermined level p has attracted some criticisms. First of all, a drawback of the traditional Value-at-Risk measure is that it does not care about the thickness of the upper tail of the distribution function from $Q_p[Y]$ on. In other words, by focusing on the VaR at a certain level, we ignore the potential severity of the default below that threshold. This means that we have no information on how bad things can become in a real stress situation. Therefore, the important question of 'how bad is bad' is left unanswered. Secondly, it is not a coherent risk measure, as suggested

by Artzner et al. (1999). More specifically, it fails to fulfil the subadditivity requirement which states that a risk measure should always reflect the advantages of diversifying, that is, a portfolio will risk an amount no more than, and in some cases less than, the sum of the risks of the constituent positions. It is possible to provide examples that show that VaR is sometimes in contradiction with this subadditivity requirement.

Artzner et al. (1999) suggested the use of Conditional VaR (CVaR) as risk measure, which they describe as a coherent risk measure. CVaR is also known as TVaR, or Tail Value-at-Risk and is defined for a level p as follows:

$$\text{TVaR}_p[Y] = \frac{1}{1-p} \int_p^1 Q_q[Y] dq. \tag{10}$$

This formula boils down to taking the arithmetic average of the quantiles of Y, from p on. This formula already makes clear that $\text{TVaR}_p[Y]$ will always be larger than the corresponding quantile.

A third risk measure is the Conditional Tail Expectation (CTE) at level p denoted by $\text{CTE}_p[Y]$ and defined as

$$\text{CTE}_p[Y] = \text{E}[Y \mid Y > Q_p[Y]], \qquad p \in (0,1). \tag{11}$$

The conditional tail expectation at level p can be seen as the mean of the top $(1-p)\%$ losses. It can also be interpreted as the VaR at level p augmented by the average exceeding of the claims Y over that quantile, given that such an exceeding occurs.
If the cumulative distribution function of Y is continuous, $\text{TVaR}_p[Y]$ equals $\text{CTE}_p[Y]$ otherwise it holds that

$$\text{CTE}_p[Y] = \text{TVaR}_{F_Y(Q_p[Y])}[Y], \qquad p \in (0,1).$$

We also introduce the Conditional Left Tail Expectation, denoted by $\text{CLTE}_p[Y]$, which for continuous random variables is defined as

$$\text{CLTE}_p[Y] = \text{E}[Y \mid Y < Q_p[Y]], \qquad p \in (0,1). \tag{12}$$

In the sequel we will be confronted with the quantiles of a function of a random variable. The following lemma will allow us to express those as a function in terms of quantiles of the random variable.

Lemma 3.1 (Quantiles transformed random variables) *Let Y be a real-valued random variable, and $0 < p < 1$. For any non-decreasing and left continuous function g, it holds that*

$$Q_p[g(Y)] = g(Q_p[Y]). \tag{13}$$

On the other hand, for any non-increasing and right continuous function g, one has

$$Q_p[g(Y)] = g(Q_{1-p}^+[Y]). \tag{14}$$

A proof of this result can be found e.g. Dhaene et al. (2002a).

As an application of Lemma 3.1, we immediately find that for any $p \in (0,1)$

$$\mathrm{E}[Y \mid Y < Q_p^+[Y]] = -\mathrm{CTE}_{1-p}[-Y], \tag{15}$$

which in case of a strictly increasing cdf F_Y is equivalent to

$$\mathrm{CLTE}_p[Y] = -\mathrm{CTE}_{1-p}[-Y]. \tag{16}$$

Corollary 3.2 *Let Y be a real-valued random variable, and $0 < p < 1$. For any non-decreasing linear function g, it holds that*

$$\mathrm{TVaR}_p[g(Y)] = g(\mathrm{TVaR}_p[Y]). \tag{17}$$

On the other hand, for any non-increasing linear function function g, one has

$$\mathrm{TVaR}_p[g(Y)] = g\left(\frac{1}{1-p} \int_p^1 Q_{1-q}^+[Y]dq\right). \tag{18}$$

4. THE HEDGING PROBLEM

Analogously to Ahn et al. (1999), Annaert et al. (2006), Deelstra et al. (2006) and Heyman et al. (2006), we assume that we have, at time zero, one financial asset and we will sell this asset at time T. When this financial asset has a maturity S, we assume that T is prior to S. In case of a decrease of its value, not hedging can lead to severe losses. Therefore, the company decides to spend an amount C on hedging. This amount will be used to buy one or part of a put option with the asset as underlying, so that, in case of a substantial decrease in the asset price, the put option can be exercised in order to prevent large losses. The remaining question now is how to choose the strike price of the option.

We will find the optimal strike price which minimizes a given risk measure for a given hedging cost. An alternative interpretation of our set up is that it can be used to calculate the minimal hedging budget the firm has to spend in order to achieve a specified risk measure level. The latter set up was followed in the paper by Miyazaki (2001) for the case of a bond and VaR.

4.1. The loss function

Let us assume that the institution has an exposure to a risky financial asset $X(> 0)$, which may have a maturity S as in the case of a bond, and that the company has decided to hedge the value by using a percentage h ($0 < h < 1$) of one put option $P(0, T, K)$ with strike price K and exercise date T (with $T \leq S$).

Further, we assume that the distribution of $X(T)$ is known and is continuous.

Analogously to Ahn et al. (1999), Annaert et al. (2006), Deelstra et al. (2006) and Heyman et al. (2006), we can look at the future value of the hedged portfolio that is composed of the asset X and the put option $P(0, T, K)$ at time T as a function of the form

$$H(T) = \max(hK + (1 - h)X(T), X(T)).$$

In a worst case scenario — a case which is of interest to us — the put option finishes in-the-money. Then the future value of the portfolio equals

$$H(T) = (1 - h)X(T) + hK.$$

Taking into account the cost of setting up our hedged portfolio, which is given by the sum of the asset price $X(0)$ and the cost C of the position in the put option, we get for the value of the loss:

$$L = X(0) + C - ((1 - h)X(T) + hK), \tag{19}$$

and this under the assumption that the put option finishes in-the-money.

When the measure $\rho[\cdot]$ satisfies the properties of translation invariance and positive homogeneity (see §3.1), we obtain

$$\rho[L] = \rho[X(0) + C - ((1 - h)X(T) + hK)] = X(0) + C - hK + (1 - h)\rho[-X(T)]. \tag{20}$$

4.2. Risk minimization

We study the case of determining the optimal strike K when minimizing the risk of L under a constraint on the hedging cost.

The risk measure of a portfolio $H = \{X, h, P\}$ consisting of a risky asset X and h put options P (which are assumed to be in-the-money at expiration) with a strike price K and an expiry date T is given by (20)[1].

We would like to minimize the risk of the future value of the hedged portfolio $H(T)$, given a maximum hedging expenditure C. More precisely, we consider the minimization problem

$$\min_{K,h} X(0) + C - hK + (1 - h)\rho[-X(T)]$$

subject to the restrictions $C = hP(0, T, K)$ and $h \in (0, 1)$.

This is a constrained optimization problem with Lagrange function

$$\mathcal{L}(K, h, \lambda) = \rho[L] - \lambda(C - hP(0, T, K)),$$

containing one multiplicator λ. Note that the multiplicators to include the inequalities $0 < h$ and $h < 1$ are zero since these constraints are not binding. Taking into account that the

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[1]In case of an unhedged portfolio, take $C = h = 0$ in (19) and (20) to obtain the loss function L with corresponding risk measure $\rho[L]$.

optimal strike K^* will differ from zero, we find from the Kuhn-Tucker conditions

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial K} = -h + h\lambda \dfrac{\partial P}{\partial K}(0, T, K) = 0 \\[2mm] \dfrac{\partial \mathcal{L}}{\partial h} = -(K + \rho[-X(T)]) + \lambda P(0, T, K) = 0 \\[2mm] \dfrac{\partial \mathcal{L}}{\partial \lambda} = -C + hP(0, T, K) = 0 \\[2mm] 0 < h < 1 \qquad \text{and} \qquad \lambda > 0 \end{cases}$$

that this optimal strike K^* should satisfy the following equation

$$P(0, T, K) - (K + \rho[-X(T)])\frac{\partial P}{\partial K}(0, T, K) = 0. \tag{21}$$

Important remarks

1. We note that the optimal strike price is independent of the hedging cost C. This independence implies that for the optimal strike K^*, $\rho[L]$ in (20) is a linear function of h (or C):

$$\rho[L] = X(0) + \rho[-X(T)] + h(P(0, T, K^*) - \rho[-X(T)] - K^*). \tag{22}$$

So, there is a linear trade-off between the hedging expenditure and the risk measure level. Although the set up of the paper is determining the strike price which minimizes a certain risk criterion, given a predetermined hedging budget, this trade-off shows that the analysis and the resulting optimal strike price can evidently also be used in the case where a firm is fixing a nominal value for the risk criterion and seeks the minimal hedging expenditure needed to achieve this risk level. It is clear that, once the optimal strike price is known, we can determine, in both approaches, the remaining unknown variable (either $\rho[L]$, either C).

2. We also note that the optimal strike price is higher than the risk measure level $-\rho[-X(T)]$. This has to be the case since $P(0, T, K)$ is always positive and the change in the price of a put option due to an increase in the strike is also positive. This result is also quite intuitive since there is no point in taking a strike price which is situated below the asset price you expect in a worst case scenario.

VaR, TVaR and CTE minimization
In this section, we focus on three particular model dependent risk measures VaR, TVaR and CTE which satisfy the properties of monotonicity, translation invariance and positive homogeneity. Recall that TVaR and CTE coincide when the cdf of the random variable of which the risk is measured is continuous.

The VaR at a level p of the loss L is minimized under the given budget constraint for the optimal strike K^* that satisfies (21) with

$$\rho[-X(T)] = Q_p[-X(T)] = -Q_{1-p}^+[X(T)], \tag{23}$$

where we used (14) from Lemma 3.1 in the last equality.

For the corresponding TVaR at level p of the loss L the optimal strike is a solution to (21) with

$$\rho[-X(T)] = \text{TVaR}_p[-X(T)] = \frac{-1}{1-p} \int_p^1 Q_{1-q}^+[X(T)]dq, \tag{24}$$

where in the last equality we invoked (18) from Corollary 3.2.

Finally, for the Conditional Tail Expectation at level p of the loss L, the optimal strike is a solution to (21) with

$$\rho[-X(T)] = \text{CTE}_p[-X(T)] = -\text{E}[X(T) \mid X(T) < Q_{1-p}^+[X(T)]], \tag{25}$$

where we applied relation (15).

When the cdf of $X(T)$ is strictly increasing, then $Q_{1-p}^+[X(T)] = Q_{1-p}[X(T)]$ in (23), (24) and according to definition (12) relation (25) becomes

$$\rho[-X(T)] = \text{CTE}_p[-X(T)] = -\text{CLTE}_{1-p}[X(T)]. \tag{26}$$

5. MULTIPLE RISKS

We consider now the case that X is not one risky asset but a sum of several risky assets as for example a basket of asset prices or a coupon-bearing bond. Thus, let us assume that

$$X = \sum_{i=1}^n a_i X_i \tag{27}$$

for some real positive constants a_i, $i = 1, \ldots, n$. Then the reasoning of the previous section remains valid. We can further elaborate the formulas under additional assumptions that are satisfied in some practical cases.

5.1. Comonotonic sum

When we assume that the components X_i in (27) are non-independent random variables and form a comonotonic vector $(X_1, \ldots, X_n)$, then X is a comonotonic sum and we can apply the additivity of comonotonic risks and the positive homogeneity property:

$$\rho[-X(T)] = \rho[-\sum_{i=1}^n a_i X_i(T)] = \sum_{i=1}^n a_i \rho[-X_i(T)]. \tag{28}$$

When we further assume that the put option is given by a discounted expectation under an appropriate measure, i.e.

$$P(0, T, K) = \text{disc} \cdot \text{E}[(K - X(T))_+], \tag{29}$$

where disc is a short hand notation for discount factor[2], we have a similar decomposition of the put option price:

$$P(0, T, K) = \sum_{i=1}^{n} a_i P_i(0, T, K_i) \quad \text{with} \sum_{i=1}^{n} a_i K_i = K, \tag{30}$$

where $P_i(0, T, K_i)$ is the put option with X_i as underlying and with maturity T and strike K_i. Kaas et al. (2000) provides a characterisation of the components K_i in the decomposition of the strike K:

$$K_i = \alpha F_{X_i(T)}^{-1}(F_{X(T)}(K)) + (1 - \alpha) F_{X_i(T)}^{-1+}(F_{X(T)}(K)) := F_{X_i(T)}^{-1(\alpha)}(F_{X(T)}(K)), \tag{31}$$

where $\alpha \in (0, 1)$ follows from $\sum_{i=1}^{n} a_i F_{X_i(T)}^{-1(\alpha)}(F_{X(T)}(K)) = K$, namely

$$\alpha = \frac{K - \sum_{i=1}^{n} a_i F_{X_i(T)}^{-1+}(F_{X(T)}(K))}{\sum_{i=1}^{n} a_i (F_{X_i(T)}^{-1}(F_{X(T)}(K)) - F_{X_i(T)}^{-1+}(F_{X(T)}(K)))} \tag{32}$$

when $F_{X_i(T)}^{-1}(F_{X(T)}(K)) \neq F_{X_i(T)}^{-1+}(F_{X(T)}(K))$ and where without loss of generality we take $\alpha = 1$ otherwise. We call the inverse in (31) the alpha-inverse cdf.

The probability measure under which the cdfs of the $X_i(T)$ are considered, is the same as the one of the expectation in the put option pricing rule but may differ from the probability measure in which the risk is considered.

The equation (21) for the optimal strike K^* of the hedging problem under consideration is now equivalent to the following set of equations:

$$\sum_{i=1}^{n} a_i P_i(0, T, K_i) - (K + \sum_{i=1}^{n} a_i \rho[-X_i(T)]) \sum_{i=1}^{n} a_i \frac{\partial P_i}{\partial K_i}(0, T, K_i) \frac{\partial K_i}{\partial K} = 0 \tag{33}$$

$$\sum_{i=1}^{n} a_i K_i = K \tag{34}$$

$$\sum_{i=1}^{n} a_i \frac{\partial K_i}{\partial K} = 1. \tag{35}$$

We can further simplify equation (33) since according to Breeden and Litzenberger (1978)

$$\frac{\partial P_i}{\partial K_i}(0, T, K_i) = \text{disc} \cdot F_{X_i(T)}(K_i). \tag{36}$$

Plugging in the expression (31) for K_i we find that this first order derivative is independent of i:

$$\frac{\partial P_i}{\partial K_i}(0, T, K_i) = \text{disc} \cdot F_{X_i(T)}(F_{X_i(T)}^{-1(\alpha)}(F_{X(T)}(K))) = \text{disc} \cdot F_{X(T)}(K). \tag{37}$$

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[2]The discount factor can be e^{-rT} with r the risk free interest rate or $\text{ZB}(0, T)$ a zero-coupon bond which pays 1 at time T.

Hence, using (35) and (37) we obtain

$$\sum_{i=1}^{n} a_i \frac{\partial P_i}{\partial K_i}(0, T, K_i)\frac{\partial K_i}{\partial K} = \text{disc} \cdot F_{X(T)}(K). \tag{38}$$

We substitute (31), (34) and (38) in (33).
Then, to find the optimal strike K^* one has to proceed as follows:

Step 1 Denote

$$A_K := F_{X(T)}(K) \tag{39}$$

and solve the following equation for A_K:

$$\sum_{i=1}^{n} a_i P_i(0, T, F_{X_i(T)}^{-1(\alpha)}(A_K)) - \text{disc} A_K \sum_{i=1}^{n} a_i (F_{X_i(T)}^{-1(\alpha)}(A_K) + \rho[-X_i(T)]) = 0. \tag{40}$$

Step 2 Plug the found value for A_K in (31) and substitute the result in (34):

$$K^* = \sum_{i=1}^{n} a_i F_{X_i(T)}^{-1(\alpha)}(A_K). \tag{41}$$

Remarks

1. As in the case of a single underlying, the optimal strike price is independent of the hedging cost and one can look at the trade-off between the hedging expenditure and the risk measure level cfr. (22).

2. In relation (28) we decomposed the risk in its components. However the derivation of (40) could also be done with $\rho[-X(T)]$ itself, leading to

$$\sum_{i=1}^{n} a_i P_i(0, T, F_{X_i(T)}^{-1(\alpha)}(A_K)) - \text{disc} A_K (\sum_{i=1}^{n} a_i F_{X_i(T)}^{-1(\alpha)}(A_K) + \rho[-X(T)]) = 0 \tag{42}$$

instead of (40).

3. Relations (40)-(41) or (41)-(42) require the knowledge of the cdf of the $X_i(T)$. Another approach is to work with the equivalent formula (21) in the multiple underlyings settings in the approximate form:

$$P(0, T, K) - (K + \rho[-X(T)])\frac{\triangle P}{\triangle K}(0, T, K) = 0, \tag{43}$$

and use option prices received from a financial institution. Of course this will lead to an approximate value for the optimal strike. On the other hand one works with the true option prices and not with the theoretical ones.

4. An application of this can be found in Annaert et al. (2006) where $X(T)$ is a coupon-bearing bond and a short-rate model with an affine term structure is assumed. In the numerical illustration Annaert et al. (2006) focus on the one-factor Hull-White model for the instantaneous-interest-rate, which is calibrated to a set of cap prices. Both VaR and TVaR are considered as risk measures.

5. When we decompose the total cost C in a similar way as the risk $X(T)$ and the option $P(0, T, K)$, i.e.

$$C = \sum_{i=1}^{n} a_i C_i,$$

then $C_i = h P_i(0, T, K_i)$ can be interpreted as the cost of the position in the put option $P_i(0, T, K_i)$ to hedge the loss incured by $X_i(T)$.

5.2. Non-comonotonic sum

We study the case that the components X_i in (27) are non-independent random variables but do not form a comonotonic vector $(X_1, \ldots, X_n)$. In that case the decomposition property (28) for the risk does not longer hold, nor does (30) for the put option price.

One could of course make use of numerical/simulation methods to solve equation (21) in the multiple risks settings or equation (43) for K.

Another approach consists in approximating the put option price by replacing the vector $(X_1, \ldots, X_n)$ by a comonotonic one, such as $(F_{X_1(T)}^{-1}(U), \ldots, F_{X_n(T)}^{-1}(U))$, see (2), with U being a uniform random variable on $(0, 1)$ or $(\mathrm{E}[X_1(T)|\Lambda], \ldots, \mathrm{E}[X_n(T)|\Lambda])$, see (7), with Λ a carefully chosen conditioning variable, and then applying the method of the previous section to it.

Let us denote

$$X_i^{\ell}(T) := \mathrm{E}[X_i(T)|\Lambda] \quad \text{and} \quad X_i^c(T) := F_{X_i(T)}^{-1}(U) \tag{44}$$

and

$$X^{\ell}(T) := \sum_{i=1}^{n} a_i X_i^{\ell}(T) \quad \text{and} \quad X^c(T) := \sum_{i=1}^{n} a_i X_i^c(T), \tag{45}$$

then $X^{\ell}(T)$ is a lower bound in convex order sense of $X(T)$ while $X^c(T)$ is an upper bound in convex order sense (see (6)) and by Definition 2.1

$$\mathrm{E}[(K - X^{\ell}(T))_+] \leq \mathrm{E}[(K - X(T))_+] \leq \mathrm{E}[(K - X^c(T))_+]. \tag{46}$$

Let us further denote the corresponding put option prices for $\nu = \ell$ and $\nu = c$ as:

$$P^{\nu}(0, T, K) = \mathrm{disc} \cdot \mathrm{E}[(K - X^{\nu}(T))_+]. \tag{47}$$

$X^c(T)$ is by its definition a comonotonic sum, however one has to choose the conditioning variable Λ very carefully such that $X^{\ell}(T)$ is also a comonotonic sum. We give more details about this choice in the application, see section 6.

Assuming that $X^\ell(T)$ is a comonotonic sum, the following decomposition of the put option price holds in view of (30)-(32):

$$P^\nu(0, T, K) = \text{disc} \cdot \sum_{i=1}^n a_i \mathrm{E}[(K_i^\nu - X_i^\nu(T))_+] := \sum_{i=1}^n a_i P_i^\nu(0, T, K_i^\nu), \quad \nu = \ell, c, \quad (48)$$

with

$$K_i^\nu = F_{X_i^\nu(T)}^{-1(\alpha)}(F_{X^\nu(T)}(K))$$

satisfying $\sum_{i=1}^n a_i K_i^\nu = K$ and with $\alpha \in (0, 1)$ given by

$$\alpha = \frac{K - \sum_{i=1}^n a_i F_{X_i^\nu(T)}^{-1+}(F_{X^\nu(T)}(K))}{\sum_{i=1}^n a_i (F_{X_i^\nu(T)}^{-1}(F_{X^\nu(T)}(K)) - F_{X_i^\nu(T)}^{-1+}(F_{X^\nu(T)}(K))}$$

when $F_{X_i^\nu(T)}^{-1}(F_{X^\nu(T)}(K)) \neq F_{X_i^\nu(T)}^{-1+}(F_{X^\nu(T)}(K))$ and with $\alpha = 1$ otherwise.

Case: $\rho[-X(t)]$ can be calculated

When it is possible to compute $\rho[-X(T)]$, we study the approximate constrained minimization problems for $\nu = \ell$ and $\nu = c$:

$$\min_{K,h} X(0) + C - hK + (1 - h)\rho[-X(T)]$$

$$\text{subject to } C = hP^\nu(0, T, K) \text{ and } h \in (0, 1). \quad (49)$$

The corresponding optimal strikes K_ν^*, $\nu = \ell, c$, are found in two steps:

Step 1 Denote

$$A_K^\nu := F_{X^\nu(T)}(K) \quad (50)$$

and solve the following equation for A_K^ν:

$$\sum_{i=1}^n a_i P_i^\nu(0, T, F_{X_i^\nu(T)}^{-1(\alpha)}(A_K^\nu)) - \text{disc} A_K^\nu (\sum_{i=1}^n a_i F_{X_i^\nu(T)}^{-1(\alpha)}(A_K^\nu) + \rho[-X(T)]) = 0. \quad (51)$$

Step 2 Plug the found value for A_K^ν in the decomposition formula of K:

$$K_\nu^* = \sum_{i=1}^n a_i F_{X_i^\nu(T)}^{-1(\alpha)}(A_K^\nu). \quad (52)$$

Case: $\rho[-X(t)]$ cannot be calculated

When it is not possible to find $\rho[-X(T)]$ we could also approximate it by $\rho[-X^\nu(T)]$, $\nu = \ell, c$ which can be decomposed according to the additivity property (28) of comonotonic risks:

$$\rho[-X^\nu(T)] = \sum_{i=1}^n a_i \rho[-X_i^\nu(T)]. \quad (53)$$

Then, we solve the approximate constrained minimization problems for $\nu = \ell$ and $\nu = c$:

$$\min_{K,h} X(0) + C - hK + (1-h)\rho[-X^\nu(T)]$$
$$\text{subject to } C = hP^\nu(0, T, K) \text{ and } h \in (0, 1).$$
(54)

The corresponding optimal strikes K_ν^*, $\nu = \ell, c$, are found in two steps:

Step 1 Denote

$$A_K^\nu := F_{X^\nu(T)}(K)$$
(55)

and solve the following equation for A_K^ν:

$$\sum_{i=1}^n a_i P_i^\nu(0, T, F_{X_i^\nu(T)}^{-1(\alpha)}(A_K^\nu)) - \text{disc}A_K^\nu \sum_{i=1}^n a_i(F_{X_i^\nu(T)}^{-1(\alpha)}(A_K^\nu) + \rho[-X_i^\nu(T)]) = 0. \quad (56)$$

Step 2 Plug the found value for A_K^ν in the decomposition formula of K:

$$K_\nu^* = \sum_{i=1}^n a_i F_{X_i^\nu(T)}^{-1(\alpha)}(A_K^\nu).$$
(57)

Based on some ordering (stochastic dominance, stop-loss order, convex order) of the risks, an ordering of the risk measures has been studied, see e.g. Dhaene et al. (2006). The mutual relation of the approximate option prices and the exact one is given by (46). However in our study we are dealing with nonlinear constrained optimization problems for which we cannot say a priori something about the mutual position of the reached constrained minima. So, it is hard to derive from the theory which approximate problem will provide a good estimate for the real optimal strike K^*. From literature we know that when the $X_i(T)$ are lognormal and their weighted sum is 'nearly comonotonic', the cdf $F_{X^c(T)}$ will be close to the cdf $F_{X(T)}$ and also the put option price $P^c(0, T, K)$ will be a good approximation for $P(0, T, K)$. Even better approximations are obtained when working with $X^\ell(T)$. Moreover in that case one has one parameter to play with, namely the conditioning variable Λ. Vanduffel et al. (2006) discuss global and local choices for it.

We will consider some numerical experiments in which we will compare the computed optimal strike K_ν^* with the optimal strike K^* obtained through simulations. We consider a coupon-bearing bond where a two-additive-factor Gaussian model (see Brigo and Mercurio (2001)) is assumed for the instantaneous interest rate. For this model an analytical expression for the put option price on the coupon-bearing bond is available, see (72).

6. APPLICATION

6.1. The two-additive-factor Gaussian model G2++

In this section we consider as an application coupon bonds in the two-additive-factor Gaussian model, shortly the G2++ model, which is an interest-rate model where the instantaneous-short-rate process is given by the sum of two correlated Gaussian factors plus a deterministic

function that is properly chosen so as to exactly fit the current term structure of discount factors. The model is quite analytically tractable in that explicit formulas for discount bonds, European options on pure discount bonds, hence caps and floors, can be readily derived.

Another consequence of the presence of two factors is that the actual variability of market rates is described in a better way: Among other improvements, a non-perfect correlation between rates of different maturities is introduced. However the non-perfect correlation especially for long maturities can be numerically non-distinguishable. For correlations of the factors close to minus one this two-factor model may degenerate.

The Gaussian model of this section is naturally related to the Hull and White (1994) two-factor model in that one can actually prove the equivalence between these two approaches. However, according to Brigo and Mercurio (2001), the formulation with two additive factors leads to less complicated formulas and is easier to implement in practice, even though we may lose some insight and intuition on the nature and the interpretation of the two factors.

The Short-Rate Dynamics

We assume that the dynamics of the instantaneous-short-rate process under the risk-adjusted measure Q is given by

$$r(t) = x(t) + y(t) + \varphi(t), \quad r(0) = r_0, \tag{58}$$

where the processes $\{x(t) : t \geq 0\}$ and $\{y(t) : t \geq 0\}$ satisfy

$$dx(t) = -ax(t)dt + \sigma d\hat{W}_1(t), \quad x(0) = 0, \tag{59}$$

$$dy(t) = -by(t)dt + \eta d\hat{W}_2(t), \quad y(0) = 0, \tag{60}$$

where $(\hat{W}_1, \hat{W}_2)$ is a two-dimensional Brownian motion with instantaneous correlation ρ:

$$d\hat{W}_1(t)d\hat{W}_2(t) = \rho dt, \tag{61}$$

where r_0, a, b, σ, η are positive constants, and where $-1 \leq \rho \leq 1$. The function φ is deterministic and well defined in the time interval $[0, T^*]$, with T^* a given time horizon, typically 10, 30 or 50 (years). In particular, $\varphi(0) = r_0$. We denote by $\mathcal{F}_t$ the sigma-field generated by the pair (x, y) up to time t.

Simple integration implies that for each $t > 0$

$$r(t) = \sigma \int_0^t e^{-a(t-u)} d\hat{W}_1(u) + \eta \int_0^t e^{-b(t-u)} d\hat{W}_2(u) + \varphi(t), \tag{62}$$

meaning that $r(t)$ (conditional on $\mathcal{F}_0$) is normally distributed.

When the term structure of discount factors that is currently observed in the market is given by the sufficiently smooth function $S \mapsto \text{ZB}^M(0, S)$, then the function φ is determined in function of the current observed bond prices $\text{ZB}^M(0, S)$ and the price at time t of a zero-coupon bond maturing at S can be written as:

$$\text{ZB}(t, S) = A(t, S) \exp[-B(a, t, S)x(t) - B(b, t, S)y(t)], \tag{63}$$

where

$$A(t, S) = \frac{\text{ZB}^M(0, S)}{\text{ZB}^M(0, t)} \exp \left\{ \frac{1}{2}[V(t, S) - V(0, S) + V(0, t)] \right\} \quad (64)$$

$$B(z, t, S) = \frac{1 - e^{-z(S-t)}}{z} \quad (65)$$

and

$$\begin{aligned}
V(t, S) = {} & \frac{\sigma^2}{a^2} \left[S - t - 2B(a, t, S) + B(2a, t, S) \right] \\
& + \frac{\eta^2}{b^2} \left[S - t - 2B(b, t, S) + B(2b, t, S) \right] \\
& + 2\rho \frac{\sigma\eta}{ab} \left[S - t - B(a, t, S) - B(b, t, S) + B(a + b, t, S) \right].
\end{aligned}$$

Hence a zero-coupon bond $\text{ZB}(t, S)$ has a lognormal distribution under Q:

$$\text{ZB}(t, S) \stackrel{d}{=} e^{\Pi^Q(0, t, S) + \Sigma(0, t, S)Z}, \quad Z \sim N(0, 1), \quad (66)$$

with

$$\Pi^Q(0, t, S) = \ln A(t, S) = \ln \left(\frac{\text{ZB}^M(0, S)}{\text{ZB}^M(0, t)} \right) + \frac{1}{2} \left[V(t, S) - V(0, S) + V(0, t) \right], \quad (67)$$

and with

$$\begin{aligned}
\Sigma(0, t, S)^2 = {} & \sigma^2 B(a, t, S)^2 B(2a, 0, t) + \eta^2 B(b, t, S)^2 B(2b, 0, t) \\
& + 2\rho\sigma\eta B(a, t, S)B(b, t, S)B(a + b, 0, t). \quad (68)
\end{aligned}$$

Besides, a zero-coupon bond $\text{ZB}(t, S)$ also has a lognormal distribution of the form (66) under the forward-neutral measure Q^T with the same variance parameter $\Sigma(0, t, S)$ but with a different mean parameter $\Pi^{Q^T}(0, t, S)$:

$$\Pi^{Q^T}(0, t, S) = \ln \left(\frac{\text{ZB}^M(0, S)}{\text{ZB}^M(0, t)} \right) - \frac{1}{2}\Sigma(0, t, S)^2, \quad (69)$$

The time zero price of a European put option with maturity T and strike K written on a zero-coupon bond with unit face value and maturity S is given by, see Brigo and Mercurio (2001):

$$\begin{aligned}
\text{ZBP}(0, T, S, K) = {} & \text{ZB}(0, T)K\Phi \left(\frac{\ln \frac{K\text{ZB}(0,T)}{\text{ZB}(0,S)}}{\Sigma(0, T, S)} + \frac{1}{2}\Sigma(0, T, S) \right) \\
& - \text{ZB}(0, S)\Phi \left(\frac{\ln \frac{K\text{ZB}(0,T)}{\text{ZB}(0,S)}}{\Sigma(0, T, S)} - \frac{1}{2}\Sigma(0, T, S) \right). \quad (70)
\end{aligned}$$

The arbitrage-free time zero price of a put option with maturity date T and strike K upon a coupon-bearing bond with price at time T

$$\mathrm{CB}(T, \mathcal{T}, \mathcal{C}) = \sum_{i=1}^{n} c_i \mathrm{ZB}(T, S_i) \tag{71}$$

with coupons $\mathcal{C} = \{c_1, \ldots, c_n\}$ paid out at times $\mathcal{T} = \{S_1, \ldots, S_n\}$ larger than T, can be derived by a similar proof as in appendix D of Chapter 4 of Brigo and Mercurio (2001) and is given by numerically computing the following one-dimensional integral:

$$\mathrm{CBP}\,(0, T, \mathcal{T}, \mathcal{C}, K)$$
$$= \mathrm{ZB}(0, T) \int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}{\sigma_x \sqrt{2\pi}} \left[K\Phi(-h_1(x, K)) - \sum_{i=1}^{n} \lambda_i(x) e^{\kappa_i(x)} \Phi(-h_{2i}(x, K)) \right] dx, \tag{72}$$

where $\Phi(\cdot)$ is the cdf of a standard normal random variable and where

$$h_1(x, K) := \frac{\hat{y}(x, K) - \mu_y}{\sigma_y \sqrt{1 - \rho_{xy}^2}} - \frac{\rho_{xy}(x - \mu_x)}{\sigma_x \sqrt{1 - \rho_{xy}^2}}$$

$$h_{2i}(x, K) := h_1(x, K) + B(b, T, S_i)\sigma_y \sqrt{1 - \rho_{xy}^2}$$

$$\lambda_i(x) := c_i A(T, S_i) e^{-B(a, T, S_i)x}$$

$$\kappa_i(x) := -B(b, T, S_i) \left[\mu_y - \frac{1}{2}(1 - \rho_{xy}^2)\sigma_y^2 B(b, T, S_i) + \rho_{xy}\sigma_y \frac{x - \mu_x}{\sigma_x} \right],$$

with $\hat{y} = \hat{y}(x, K)$ the solution of the following equation for each fixed x

$$\sum_{i=1}^{n} c_i A(T, S_i) e^{-B(a, T, S_i)x - B(b, T, S_i)\hat{y}} = K \tag{73}$$

and

$$\sigma_x := \sigma \sqrt{B(2a, 0, T)}, \qquad \sigma_y := \eta \sqrt{B(2b, 0, T)}, \tag{74}$$

$$\rho_{xy} := \frac{\rho\sigma\eta}{\sigma_x \sigma_y} B(a + b, 0, T) \tag{75}$$

$$\mu_x := -\left(\frac{\sigma^2}{a} + \rho\frac{\sigma\eta}{b}\right) B(a, 0, T) + \frac{\sigma_x^2}{a} + \frac{\rho_{xy}\sigma_x\sigma_y}{b}$$

$$\mu_y := -\left(\frac{\eta^2}{b} + \rho\frac{\sigma\eta}{a}\right) B(b, 0, T) + \frac{\sigma_y^2}{b} + \frac{\rho_{xy}\sigma_x\sigma_y}{a}.$$

6.2. VaR, TVaR and CTE minimization

For the interest rate model under consideration, $X(T)$ in (27) equals $\mathrm{CB}(T, \mathcal{T}, \mathcal{C})$, see (71), and is in view of (63) a sum of lognormal random variables.

First we obtain the optimal strike K^* by solving numerically the with (21) equivalent formula for multiple risks:

$$\mathrm{CBP}(0, T, \mathcal{T}, \mathcal{C}, K) - (K + \rho[-\mathrm{CB}(T, \mathcal{T}, \mathcal{C})]) \frac{\partial \mathrm{CBP}}{\partial K}(0, T, \mathcal{T}, \mathcal{C}, K) = 0, \qquad (76)$$

where the first order derivative w.r.t. K is via standard calculations found to be:

$$\frac{\partial \mathrm{CBP}}{\partial K}(0, T, \mathcal{T}, \mathcal{C}, K) = \mathrm{ZB}(0, T) \int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}{\sigma_x \sqrt{2\pi}} \Phi(-h_1(x, K)) dx. \qquad (77)$$

The put option price (72) and its derivative (77) have a common part, by which (76) can be simplified to

$$\mathrm{ZB}(0, T) \int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}{\sigma_x \sqrt{2\pi}} \left[\rho[-\mathrm{CB}(T, \mathcal{T}, \mathcal{C})] \Phi(-h_1(x, K)) \right.$$
$$\left. + \sum_{i=1}^{n} \lambda_i(x) e^{\kappa_i(x)} \Phi(-h_{2i}(x, K)) \right] dx = 0.$$

To calculate $\rho[-\mathrm{CB}(T, \mathcal{T}, \mathcal{C})]$ for VaR or TVaR(=CTE), we simulate the cdf of the coupon-bearing bond as sum of lognormals for which the correlations can be computed, see (85).

This optimal strike K^* will be used as bench mark for strikes found as solution to the approximated problems (49) respectively (54). For these last two methods we need explicit expressions for the cdfs and the put option prices in terms of the model parameters.

Hereto we note that for $\nu = c$ and $\nu = \ell$, $X_i^\nu(T) = \mathrm{ZB}^\nu(T, S_i)$ is a continuous lognormal random variable of which the cdf $F_{\mathrm{ZB}^\nu(T, S_i)}$ is strictly increasing, hence the alpha-inverse in (31) coincides with the standard inverse, and for $x \in [0, +\infty[$

$$F_{\mathrm{ZB}^\nu(T, S_i)}(x) = \begin{cases} F_{\mathrm{ZB}(T, S_i)}(x) & \text{if } \nu = c & (78a) \\ F_{\mathrm{E}^\mathbb{Q}[\mathrm{ZB}(T, S_i)|\Lambda]}(x) & \text{if } \nu = \ell, & (78b) \end{cases}$$

with $\mathbb{Q} = Q$ or Q^T.

It is well-known that for a lognormal distributed random variable with parameters $\Pi^\mathbb{Q}(0, T, S_i)$, with $\mathbb{Q} = Q$ or Q^T, and $\Sigma(0, T, S_i)$ the inverse cdf is given by:

$$F^{-1}_{\mathrm{ZB}(T, S_i)}(p) = e^{\Pi^\mathbb{Q}(0, T, S_i) + \Sigma(0, T, S_i)\Phi^{-1}(p)}, \qquad p \in [0, 1], \qquad (79)$$

with $\Phi(\cdot)$ the cdf of the standard normal random variable.

Further, the following property based on Lemma 3.1 holds, see Dhaene et al. (2002a): for $p \in (0, 1)$

$$F^{-1}_{\mathrm{E}^\mathbb{Q}[\mathrm{ZB}(T, S_i)|\Lambda]}(p) = \mathrm{E}^\mathbb{Q}[\mathrm{ZB}(T, S_i)|\Lambda = F_\Lambda^{-1}(q)], \qquad (80)$$

with

$$q = \begin{cases} p & \text{if } \mathrm{E}^\mathbb{Q}[\mathrm{ZB}(T, S_i)|\Lambda] \text{ non-decreasing in } \Lambda & (81a) \\ 1 - p & \text{if } \mathrm{E}^\mathbb{Q}[\mathrm{ZB}(T, S_i)|\Lambda] \text{ non-increasing in } \Lambda. & (81b) \end{cases}$$

In what follows we will focus on the case that $\mathrm{E}^{\mathbb{Q}}[\mathrm{ZB}(T, S_i)|\Lambda]$ is a non-decreasing function of Λ since for the model under consideration and the choices of Λ this will be the case. The formulas for the other case are completely analogue. The different choices for the conditioning variable Λ (see below) will depend on the measure $\mathbb{Q}$, where $\mathbb{Q} = Q$ or $\mathbb{Q} = Q^T$. Therefore we already make this dependence clear in the notation from here on.

When the conditioning random variable $\Lambda^{\mathbb{Q}}$ is normally distributed with mean $\mu_{\Lambda^{\mathbb{Q}}}$ and variance $\sigma^2_{\Lambda^{\mathbb{Q}}}$, the random variable $\mathrm{ZB}(T, S_i)$ conditionally on $\Lambda^{\mathbb{Q}} = \lambda$ remains a lognormal random variable and the conditional expectation is:

$$\mathrm{E}^{\mathbb{Q}}[\mathrm{ZB}(T, S_i)|\Lambda^{\mathbb{Q}} = F_{\Lambda^{\mathbb{Q}}}^{-1}(p)] = e^{\Pi^{\mathbb{Q}}(0,T,S_i)+r_i^{\mathbb{Q}}\Sigma(0,T,S_i)\Phi^{-1}(p)+\frac{1}{2}(1-(r_i^{\mathbb{Q}})^2)\Sigma(0,T,S_i)^2}, \quad p \in (0,1), \tag{82}$$

where we used $F_{\Lambda^{\mathbb{Q}}}^{-1}(p) = \mu_{\Lambda^{\mathbb{Q}}} + \sigma_{\Lambda^{\mathbb{Q}}}\Phi^{-1}(p)$ and with $r_i^{\mathbb{Q}}$ the correlation between $\Lambda^{\mathbb{Q}}$ and $\ln(\mathrm{ZB}(T, S_i))$ (see (66)). When these correlations $r_i^{\mathbb{Q}}$ are all positive, it is clear from (82) that for all i $\mathrm{E}^{\mathbb{Q}}[\mathrm{ZB}(T, S_i)|\Lambda^{\mathbb{Q}}]$ will be non-decreasing in $\Lambda^{\mathbb{Q}}$ and that hence by Property 2.1 the sum $X^{\ell}(T) = \sum_{i=1}^{n} c_i \mathrm{ZB}^{\ell}(T, S_i)$ will be comonotonic.

The price of a put option with the lognormal $\mathrm{ZB}^{\nu}(T, S_i)$ as underlying and with maturity T and strike K can be derived from (70). For $\nu = \ell$, $\Sigma(0, T, S_i)$ has to be replaced by $r_i^{\mathbb{Q}}\Sigma(0, T, S_i)$.

For the case that $\rho[-X(T)]$ is not calculated/simulated, we have to compute in addition the risk measures VaR and TVaR — CTE equals in this case TVaR — of a lognormal random variable like appearing in relation (56). General expressions for it can be found in several papers see e.g. Annaert et al. (2006), Dhaene et al. (2006), Vanduffel et al. (2005). We state them here explicitly in terms of the model parameters.

Combining the definition (8) of the Value-at-Risk at level p with (14) of Lemma 3.1 and expressing the resulting quantile in terms of the inverse cdf while recalling that the cdf is strictly increasing for a lognormal random variable, we successively obtain for $p \in (0, 1)$:

$$\begin{aligned} \rho[-X_i^{\nu}(T)] = \mathrm{VaR}_p[-\mathrm{ZB}^{\nu}(T, S_i)] &= Q_p[-\mathrm{ZB}^{\nu}(T, S_i)] \\ &= -Q_{1-p}^{+}[\mathrm{ZB}^{\nu}(T, S_i)] \\ &= -F_{\mathrm{ZB}^{\nu}(T,S_i)}^{-1}(1-p). \end{aligned}$$

Finally, for $\nu = c$ invoke (79) while for $\nu = \ell$ apply (82) with $\mathbb{Q} = Q$.

The Tail Value-at-Risk for a level p is defined in (10). We have to combine it with (18) in Corollary 3.2 and write the integrand in terms of the cdf of $\mathrm{ZB}^{\nu}(T, S_i)$:

$$\begin{aligned} \rho[-X_i^{\nu}(T)] = \mathrm{TVaR}_p[-\mathrm{ZB}^{\nu}(T, S_i)] &= \frac{1}{1-p}\int_p^1 Q_q[-\mathrm{ZB}^{\nu}(T, S_i)]dq \\ &= -\frac{1}{1-p}\int_p^1 Q_{1-q}^{+}[\mathrm{ZB}^{\nu}(T, S_i)]dq \\ &= -\frac{1}{1-p}\int_p^1 F_{\mathrm{ZB}^{\nu}(T,S_i)}^{-1}(1-q)dq. \end{aligned}$$

We now invoke the expression (79) respectively (82) with $\mathbb{Q} = Q$ for the cdf, and evaluate

the resulting integral:

$$\text{TVaR}_p[-\text{ZB}^c(T,S_i)] = -\frac{e^{\Pi^Q(0,T,S_i)}}{1-p}\int_p^1 e^{\Sigma(0,T,S_i)\Phi^{-1}(1-q)}dq$$

$$= -\frac{e^{\Pi^Q(0,T,S_i)+\frac{1}{2}\Sigma(0,T,S_i)^2}}{1-p}\Phi(\Phi^{-1}(1-p)-\Sigma(0,T,S_i)),$$

respectively

$$\text{TVaR}_p[-\text{ZB}^\ell(T,S_i)] = -\frac{e^{\Pi^Q(0,T,S_i)+\frac{1}{2}(1-(r_i^Q)^2)\Sigma(0,T,S_i)^2}}{1-p}\int_p^1 e^{r_i^Q\Sigma(0,T,S_i)\Phi^{-1}(1-q)}dq$$

$$= -\frac{e^{\Pi^Q(0,T,S_i)+\frac{1}{2}\Sigma(0,T,S_i)^2}}{1-p}\Phi(\Phi^{-1}(1-p)-r_i^Q\Sigma(0,T,S_i)).$$

Now we will discuss different choices for the (normally distributed) conditioning random variable $\Lambda^{\mathbb{Q}}$ for the case where $\nu = \ell$. Note that $\Lambda^{\mathbb{Q}}$ enters the formulas via the correlations $r_i^{\mathbb{Q}}$. We will report the corresponding expressions for these correlations.

We differentiate between conditioning variables that are global and that are local. Global here means that they lead to lower bounds $X^\ell(T)$ (in convex order) for which the cdf presents a global goodness-of-fit when compared to the cdf of the original sum $X(T)$. When only a part of the distribution has to be well-fitted as for example the tails of the cdf, we can work with lower bound approximations which are locally optimal. Various choices have been proposed in literature, see e.g. Dhaene et al. (2002b), Deelstra et al. (2004), Vanduffel et al. (2005), Vanduffel et al. (2006).

We define the conditioning random variable $\Lambda^{\mathbb{Q}}$ as a linear combination of $Z_1, Z_2, \ldots, Z_n$, with Z_i showing up in the distribution of $\text{ZB}(T, S_i)$ see (66) and with the random vector $(Z_1, Z_2, \ldots, Z_n)$ having a multivariate standard normal distribution:

$$\Lambda^{\mathbb{Q}} = \sum_{i=1}^n \gamma_i^{\mathbb{Q}}\Sigma(0,T,S_i)Z_i. \tag{83}$$

Then its correlation with $\ln(\text{ZB}(T, S_i))$ is given by:

$$r_i^{\mathbb{Q}} = \text{corr}(\ln(\text{ZB}(T,S_i)), \Lambda^{\mathbb{Q}}) = \frac{\sum_{j=1}^n \gamma_j^{\mathbb{Q}}\Sigma(0,T,S_j)\text{cov}(Z_i,Z_j)}{\sqrt{\sum_{i=1}^n\sum_{j=1}^n \gamma_i^{\mathbb{Q}}\gamma_j^{\mathbb{Q}}\Sigma(0,T,S_i)\Sigma(0,T,S_j)\rho_{ij}}}, \tag{84}$$

with in view of (66)

$$\rho_{ij} = \text{corr}(Z_i, Z_j) = \text{cov}(Z_i, Z_j) = \frac{\text{cov}(\ln\text{ZB}(T,S_i), \ln\text{ZB}(T,S_j))}{\Sigma(0,T,S_i)\Sigma(0,T,S_j)}, \tag{85}$$

where by means of the relations (63), (65), (74) and (75)

$$\begin{aligned}
&\text{cov}(\ln\text{ZB}(T,S_i), \ln\text{ZB}(T,S_j)) \\
&= B(a,T,S_i)B(a,T,S_j)\sigma_x^2 + B(b,T,S_i)B(b,T,S_j)\sigma_y^2 \\
&\quad + [B(a,T,S_i)B(b,T,S_j) + B(a,T,S_j)B(b,T,S_i)]\sigma_x\sigma_y\rho_{xy}.
\end{aligned} \tag{86}$$

In particular for $i = j$, (86) equals $\Sigma(0, T, S_i)^2$ or $\Sigma(0, T, S_j)^2$, which is in accordance with (68).

By specifying the coefficients $\gamma_i^{\mathbb{Q}}$ in (83) we get different choices for $\Lambda^{\mathbb{Q}}$ and by substituting those expressions for the $\gamma_i^{\mathbb{Q}}$'s in (84) we get the corresponding correlations:

1. Taylor-based: notation TB and $\gamma_i^{\mathbb{Q}} = c_i e^{\Pi^{\mathbb{Q}}(0,T,S_i)}$

2. Geometric average based: notation GA and

$$\gamma_i^{\mathbb{Q}} = \frac{c_i}{\sqrt{\sum_{i=1}^n \sum_{j=1}^n c_i c_j \Sigma(0, T, S_i) \Sigma(0, T, S_j) \rho_{ij}}}$$

3. Maximal variance: notation MV and $\gamma_i^{\mathbb{Q}} = c_i e^{\Pi^{\mathbb{Q}}(0,T,S_i) + \frac{1}{2}\Sigma(0,T,S_i)^2}$

4. Maximal CTE and minimal CLTE: notation MCTE and

$$\gamma_i^{\mathbb{Q}} = c_i e^{\Pi^{\mathbb{Q}}(0,T,S_i) + \frac{1}{2}\Sigma(0,T,S_i)^2} \cdot e^{\frac{1}{2}(\Phi^{-1}(1-p) - r_i^{\mathbb{Q},\mathrm{MV}} \Sigma(0,T,S_i))^2},$$

where $r_i^{\mathbb{Q},\mathrm{MV}}$ denotes the correlation (84) between $\ln(\mathrm{ZB}(T, S_i))$ and MV and with p the level of the risk measure $\mathrm{CTE}_p[-X(T)] = \mathrm{TVaR}_p[-X(T)]$.

TB, GA and MV are global conditioning variables while MCTE is a local conditioning variable.

Numerical results will be presented.

References

D.H. Ahn, J. Boudoukh, M. Richardson, and R. Whitelaw. Optimal risk management using options. *Journal of Finance*, 54:359–375, 1999.

J. Annaert, G. Deelstra, D. Heyman, and M. Vanmaele. Risk management of a bond portfolio using options. *Insurance: Mathematics and Economics*, 2006. accepted.

P. Artzner, F. Delbaen, J.M. Eber, and D. Heath. Coherent measures of risk. *Mathematical Finance*, 9(3):203–229, 1999.

D.T. Breeden and R.H. Litzenberger. Prices of state-contingent claims implicit in option prices. *Journal of Business*, 51:621–651, 1978.

D. Brigo and F. Mercurio. *Interest Rate Models — Theory and Practice.* Springer Verlag, Berlin, 2001.

G. Deelstra, J. Liinev, and M. Vanmaele. Pricing of arithmetic basket options by conditioning. *Insurance: Mathematics and Economics*, 34:55–77, 2004.

G. Deelstra, A. Ezzine, D. Heyman, and M. Vanmaele. Managing Value-at-Risk for a bond using bond put options. *Computational Economics*, 2006. accepted.

J. Dhaene, M. Denuit, M.J. Goovaerts, R. Kaas, and D. Vyncke. The concept of comonotonicity in actuarial science and finance: Theory. *Insurance: Mathematics and Economics*, 31(1):3–33, 2002a.

J. Dhaene, M. Denuit, M.J. Goovaerts, R. Kaas, and D. Vyncke. The concept of comonotonicity in actuarial science and finance: Application. *Insurance: Mathematics and Economics*, 31(2):133–161, 2002b.

J. Dhaene, S. Vanduffel, Q. Tang, M.J. Goovaerts, R. Kaas, and D. Vyncke. Solvency capital, risk measures and comonotonicity: a review. Technical Report OR 0416, K.U.Leuven, 2003.

J. Dhaene, S. Vanduffel, Q. Tang, M.J. Goovaerts, R. Kaas, and D. Vyncke. Risk measures and comonotonicity: a review. *Stochastic models*, 22:573–606, 2006.

D. Heyman, J. Annaert, G. Deelstra, and M. Vanmaele. Minimization of the (Conditional) Value-at-Risk for a coupon bearing bond using a bond put option. In M. Vanmaele, A. De Schepper, J. Dhaene, H. Reynaerts, W. Schoutens, and P. Van Goethem, editors, *Handelingen Contactforum 4th Actuarial and Financial Mathematics Day, 10 februari 2006*, pages 85–96, Brussel, 2006. Koninklijke Vlaamse Academie van België voor Wetenschappen en Kunsten.

J. Hull and A. White. Numerical procedures for implementing term structure models ii: two-factor models. *The Journal of Derivatives*, 2:37–47, 1994.

R. Kaas, J. Dhaene, and M. Goovaerts. Upper and lower bounds for sums of random variables. *Insurance: Mathematics and Economics*, 27(2):151–168, 2000.

K. Miyazaki. Dual analysis on hedging VaR of bond portfolio using options. Technical report, University of Electro-Communications, Tokyo, 2001. URL `http://www.gloriamundi.org/picsresources/km.pdf`.

S. Vanduffel, T. Hoedemakers, and J. Dhaene. Comparing approximations for risk measures of sums of non-independent lognormal random variables. *North American Actuarial Journal*, 9(4):71–82, 2005.

S. Vanduffel, X. Chen, J. Dhaene, M.J. Goovaerts, L. Henrard, and R. Kaas. A note on optimal lower bound approximations for risk measures of sums of lognormals. *Journal of Computational and Applied Mathematics*, 2006. submitted.